6 Solving the wave equation for the infinite string

In this lecture I assume that my string (or rod) are so long that it is reasonable to disregard the boundary conditions, i.e., I consider an infinite one dimensional space. In this case I get the initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad t > 0, \quad -\infty < x < \infty, \tag{6.1}$$

with the initial conditions

$$u(0, x) = f(x), \quad u_t(0, x) = g(x), \quad -\infty < x < \infty. \tag{6.2}$$

Arguably the best way to get an intuitive understanding what is modeled with this equation is to imagine an infinite guitar string, where $u(t, x)$ represent a transverse displacement at time t at position x.

Very surprisingly (do not get used to it, this is a very rare case for PDE), problem (6.1)–(6.2) can be solved explicitly, and the final result if a relatively simple to use formula.

6.1 The general solution to the wave equation

First I will find the *general solution* to (6.1), i.e., the formula that includes *all* possible solutions to the wave equation. To do this I note that I can rewrite (6.1) as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Denoting

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u,$$

I find that my wave equation (6.1) is equivalent to two first order linear PDE:

$$\begin{aligned} v_t - c v_x &= 0, \\ u_t + c u_x &= v. \end{aligned}$$

From the previous lectures we know immediately that the first one has the general solution $v(t, x) = F^*(x + ct)$, for some arbitrary F^*, and the second one has the solution

$$u(t, x) = \int_0^t v(s, x - c(t - s))\mathrm{d}s + G^*(x - ct) = \int_0^t F^*(x - ct + 2cs)\mathrm{d}s + G^*(x - ct).$$

Making the change of the variables $\tau = x - ct + 2cs$ in the integral, I have

$$u(t, x) = \frac{1}{2c}\int_{x-ct}^{x+ct} F^*(\tau)\mathrm{d}\tau + G^*(x - ct). \tag{6.3}$$

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Finally, since F^*, G^* are arbitrary, by denoting

$$F(x+ct)=\frac{1}{2c}\int_0^{x+ct}F^*(\tau)\mathrm{d}\tau,\quad G(x-ct)=\frac{1}{2c}\int_{x-ct}^{0}F^*(\tau)\mathrm{d}\tau+G^*(x-ct),$$

I get the final result that

$$u(t,x)=F(x+ct)+G(x-ct),$$

for arbitrary $\mathcal{C}^{(2)}$ functions F and G. This expression, and the analysis from previous lectures, tell me that the general solution to the wave equation is a sum of two linear traveling waves, one of which moving to the left and another one moving to the right.

6.2 d'Alembert's formula

Here I will solve problem (6.1)-(6.2) and reproduce a famous formula, first obtained by Jean-Baptiste le Rond d'Alembert in 1747.

From the first condition in (6.2) I immediately get that $G^*=f$. To use the second condition I calculate

$$u_t(0,x)=F^*(x)-cf'(x)=g(x)\implies F^*(x)=g(x)+cf'(x).$$

This yields

$$u(t,x)=\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\mathrm{d}s+\frac{1}{2}f(x+ct)-\frac{1}{2}f(x-ct)+f(x-ct),$$

and finally

$$u(t,x)=\frac{1}{2}\Big(f(x+ct)+f(x-ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\mathrm{d}s, \tag{6.4}$$

which is d'Alembert's formula.

Exercise 1. All the results above, including the general solution and d'Alembert's formula, can be obtained in a different way, without the trick I used to factor the differential operator $(\partial_{tt}-c^2\partial_{xx})$. Namely, show that the change of variables

$$\eta=x+ct,\quad \xi=x-ct$$

reduces the wave equation to its *canonical* form

$$v_{\eta\xi}=0,$$

and derive from it the general solution. Using the initial conditions and the found general solution obtain d'Alembert's formula.

Exercise 2. What are the conditions on f,g in d'Alembert's formula to guarantee that it provides a classical solution to the wave equation?

To get an intuitive understanding how formula (6.4) works consider two examples. I assume that $c=1$ below.

Example 6.1. Let my initial condition be such that the initial velocity g is zero. Then my formula simplifies to

$$u(t,x)=\frac{1}{2}\Big(f(x+ct)+f(x-ct)\Big),$$

and hence my solution is a sum of two identical linear waves, each of which is exactly half of the initial displacement. Therefore I can envision the behavior of solutions by first dividing in half the initial displacement, then shifting one half to the left and the other one to the right, and then adding them.

Let, e.g, f be defined as

$$f(x)=\begin{cases}0, & x<-0.5,\\ 1, & -0.5\leq x\leq 0.5,\\ 0, & x>0.5.\end{cases}$$

The form of the solution at different time moments is shown in Fig. 1.

Now it should be clear how the three dimensional surface of the solution naturally appears (see Fig. 2, left panel).

Finally, the behavior of solution should be also clear from looking at the plane (t,x) and several straight lines of the form

$$x=ct+\xi,\quad x=-ct+\eta,$$

which are also called the *characteristics* of the wave equation. Note that the signal spreads along the characteristics (Fig. 2, right panel).

If one is interested in actual formulas for the solution, the right panel in Fig. 2 can be used. Let me redraw it including some additional information (see Fig. 3)

On this figure I drew the initial condition along the line $t=0$, such that $f(x)=1$ where I have the bold black interval and zero everywhere else. I also included the characteristics (bold gray lines) that start at exactly the points where the analytical expression for my initial condition changes. Note that these characteristics divide the whole half-plane in six domains (denoted by roman letters). For each of these domains I will get a different expression for u. Indeed, consider, e.g., point A that belongs to domain I. Note that the two characteristics that connect the initial condition with this point are the dashed line segments in the figure. Just looking at my figure I can see that both of these characteristics end up at the part of the initial condition where $f(x)=1$, and therefore at the point A (as well as at any other point in this domain), the value of u is given by

$$u(t,x)=\frac{1+1}{2}\,,$$

where the first 1 comes from one characteristic and the second one comes from the other one.

Similarly, at point B (domain V), I must trace both characteristics back to the initial conditions, but the difference now is that one of the characteristics ends up at the the initial condition where $f(x)=1$ and the other one at the point $f(x)=0$, therefore, in this domain

$$u(t,x)=\frac{1+0}{2}\,.$$

To summarize, for this example I get the final result

$$u(t,x)=\begin{cases}1, & (t,x)\in \mathrm{I},\\ 0, & (t,x)\in \mathrm{II},\\ 0.5, & (t,x)\in \mathrm{III},\\ 0, & (t,x)\in \mathrm{IV},\\ 0.5, & (t,x)\in \mathrm{V},\\ 0, & (t,x)\in \mathrm{VI},\end{cases}$$

see again Fig. 2 to reconcile this analytical solution with the geometric picture for the surface u.

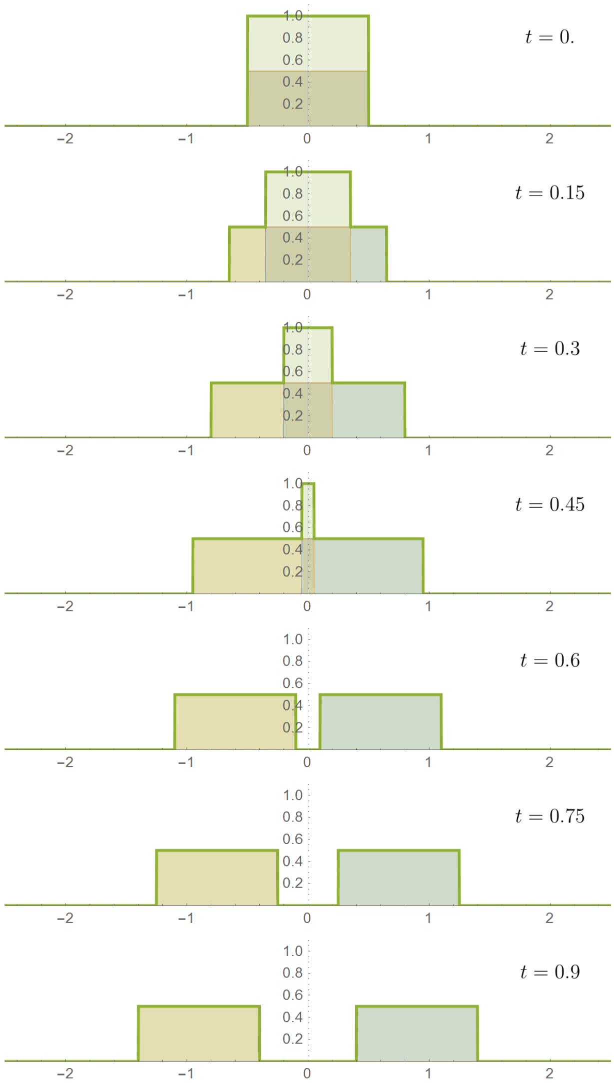

Figure 1: Solution to the initial value problem for the wave equation in case when the initial velocity is zero for different time moments. The actual solution has the bold green border.

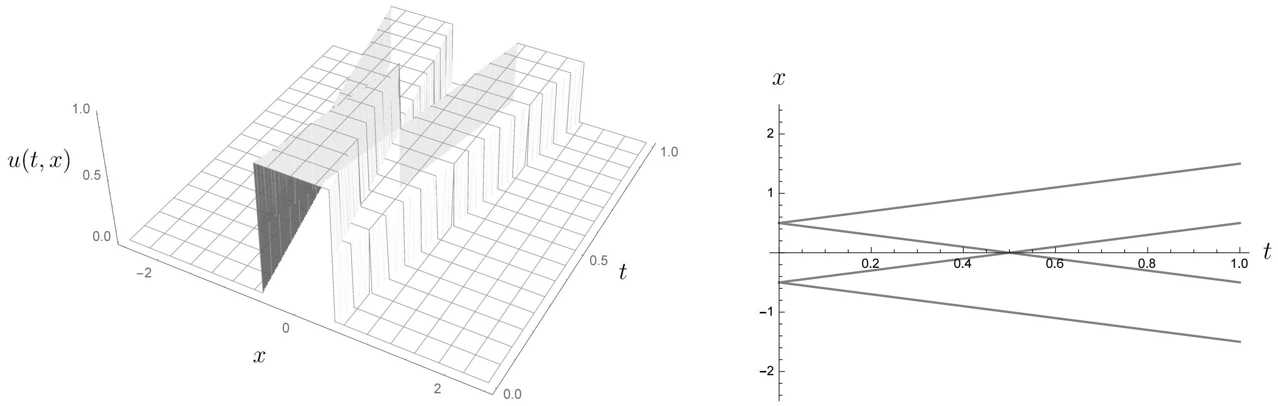

Figure 2: Left: Solution in the coordinates (t, x) to the initial value problem for the wave equation in case when the initial velocity is zero. Right: Characteristics of the wave equation on the plane (t, x).

Example 6.2. For the second example I take $f(x) = 0$ and

$$g(x) = \begin{cases} 0, & x < -0.5, \\ 1, & -0.5 \leq x \leq 0.5, \\ 0, & x > 0.5. \end{cases}$$

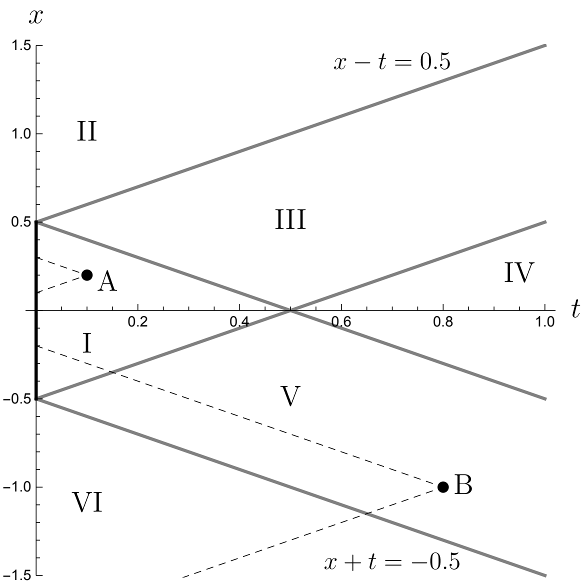

Figure 3: Calculating the analytical solution (see the text for details).

Now the solution takes the form

$$u(t,x)=\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\mathrm{d}s=\frac{1}{2c}\Big(G(x+ct)-G(x-ct)\Big),$$

where G is any antiderivative of g, e.g., I can always take

$$G(x)=\int_{-\infty}^{x}g(s)\mathrm{d}s.$$

Therefore the solution now is a difference of two traveling waves, each of which is exactly half of G. In my case

$$G(x)=\begin{cases}0, & x<-0.5,\\ x+0.5, & -0.5\leq x\leq 0.5,\\ 1, & x>0.5,\end{cases}$$

and my solution is given in Fig. 4. See also Fig. 5 for the three dimensional picture. Again the overall picture can be figured out from the plane (t,x) and characteristics on it.

Similarly to what I did for the first example, I can use Fig. 3 to obtain the analytical expressions for my solution. The difference now is that I need not add but integrate along the initial condition line. The solution in general is given by (recall that $c=1$)

$$u(t,x)=\frac{1}{2}\int_{x-t}^{x+t}g(s)\mathrm{d}s,$$

but for the specific computations the lower and upper limits may change if g is given by different analytical formulas on different intervals, and especially if g is zero for some intervals. To clarify, let me calculate the solution at point B in Fig. 3 (as well as at any other point in domain V). Note that the lower characteristic connects to the line $t=0$ at the point where $g(x)=0$, therefore, one does not need to integrate anything until the point $x=-0.5$:

$$u(t,x)=\frac{1}{2}\int_{-0.5}^{x+t}\mathrm{d}s=\frac{1}{2}(x+t+0.5)=\frac{1}{4}+\frac{x+t}{2}\,.$$

For point A I have

$$u(t,x)=\frac{1}{2}\int_{x-t}^{x+t}\mathrm{d}s=t,$$

since the characteristics connecting this point with the initial condition both end up in the interval where $g(x)=1$.

To summarize, for this second example of nonzero initial velocity I obtain

$$u(t,x)=\begin{cases}t, & (t,x)\in \mathrm{I},\\ 0, & (t,x)\in \mathrm{II},\\ \frac{1}{4}-\frac{x-t}{2}, & (t,x)\in \mathrm{III},\\ \frac{1}{2}, & (t,x)\in \mathrm{IV},\\ \frac{1}{2}+\frac{x+t}{2}, & (t,x)\in \mathrm{V},\\ 0, & (t,x)\in \mathrm{VI},\end{cases}$$

see again Fig. 5 to reconcile this analytical solution with the geometric picture for the surface u.

These two examples give a general idea how actually solutions to the wave equation behave. Note that in both cases I used the initial conditions that are not continuously differential (they are not even continuous!) and hence my solutions are definitely not the classical solutions. However, the notion of the solution to the wave equation can be extended in a way to include these, nondifferential, solutions. They are usually called *weak solutions*.

Another point to note that characteristics of the wave equation allows immediately to see which initial conditions contribute to the solution at a given point (t, x) (this is called *domain of dependence*)

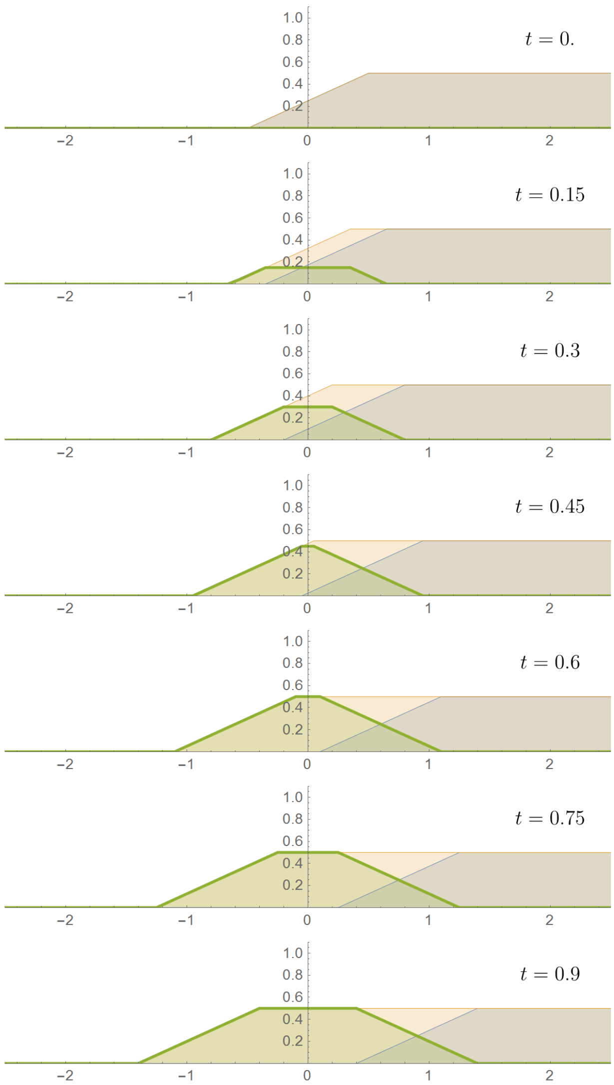

Figure 4: Solution for different time moments to the initial value problem for the wave equation in the case when the initial displacement is zero. The actual solution is shown with the green bold border. The light brown and darker brown areas represent $G(x + ct)$ and $G(x - ct)$ respectfully (see the text for the definition of G).

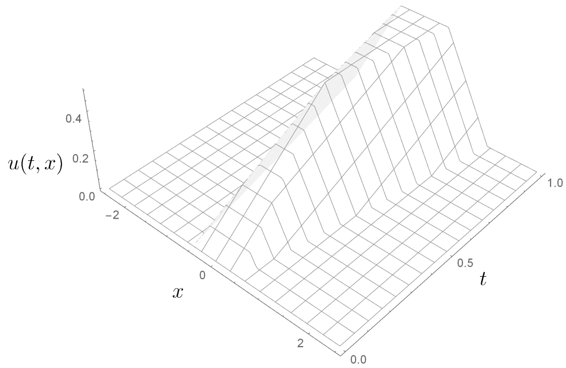

Figure 5: Solution in the coordinates (x, t) to the initial value problem for the wave equation in the case when the initial displacement is zero.

and also how the given point ξ on the initial condition spreads the signal with time (*range of influence*), see Fig. 6.

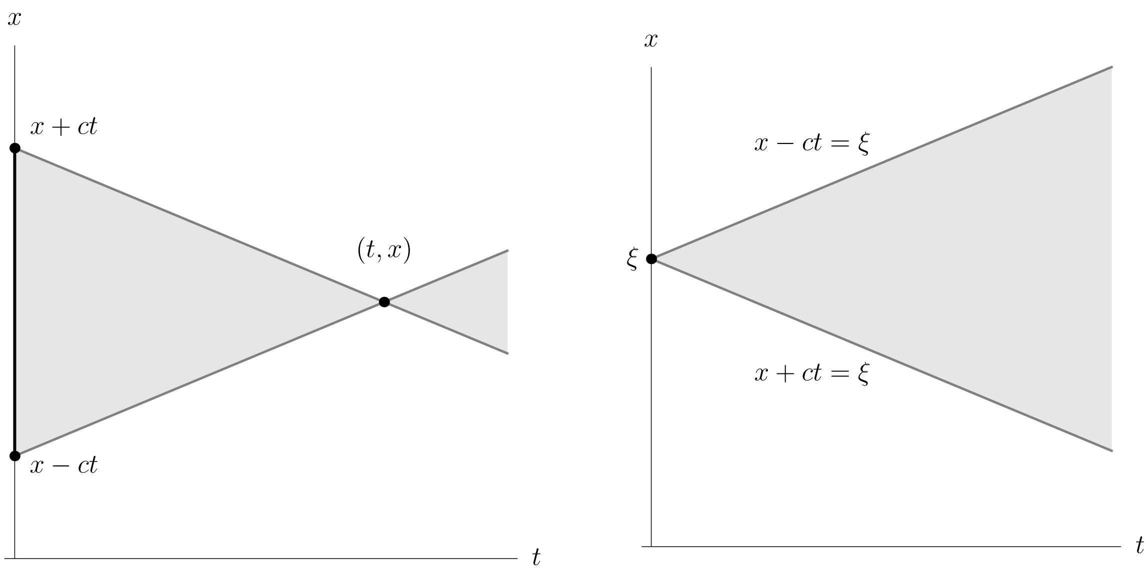

Figure 6: Domain of dependence (left, the bold line on the x axis) of the point (t, x) and the range of influence (right, the shaded area) of the point ξ.

Exercise 3. Show that the initial value problem for the wave equation on the whole real line is a well-posed problem.

6.3 Test yourself

6.1. What is the general solution to $u_{tt} = 4u_{xx}$?

6.2. Solve $u_{tt} = 2u_{xx}$, $u(0,x) = \sin x$, $u_t(0,x) = x$.

6.3. Let u solve $u_{tt} - 9u_{xx} = 0$. What is the range of influence of the point $\xi = 5$ at $t = 0$? What is the domain of dependence of the point $(5,1)$?

6.4 Solutions to the exercises

Exercise 1. The first part of this exercise is another instance of practicing the chain rule, which is very helpful in our course. So, let me start with the relations

$$u(t,x) = u\Big(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2}\Big) = v(\eta,\xi) = v(x+ct, x-ct).$$

Now, using concise yet hopefully clear notation,

$$\begin{aligned} u_t &= v_\eta \eta_t + v_\xi \xi_t = cv_\eta - cv_\xi, \\ u_{tt} &= (u_t)_t = c(v_\eta)_t - c(v_\xi)_t = c(cv_{\eta\eta} - cv_{\eta\xi} - cv_{\xi\eta} + cv_{\xi\xi}) = c^2(v_{\eta\eta} - 2v_{\eta\xi} + v_{\xi\xi}), \\ u_x &= v_\eta + v_\xi, \\ u_{xx} &= v_{\eta\eta} + 2v_{\eta\xi} + v_{\xi\xi}, \end{aligned}$$

therefore,

$$u_{tt} - c^2 u_{xx} = -4c^2 v_{\eta\xi} = 0$$

as required.

Recalling from one of the first lectures that $v_{\eta\xi} = 0$ implies that $v(\eta,\xi) = F(\eta) + G(\xi)$ for some arbitrary $\mathcal{C}^{(2)}$ functions F, G, I obtain the same general solution

$$u(t,x) = F(x+ct) + G(x-ct).$$

Using the initial conditions I have

$$F(x) + G(x) = f(x), \quad cF'(x) - cG'(x) = g(x),$$

which implies

$$F(x) + G(x) = f(x), \quad F(x) - G(x) = \frac{1}{c}\int_0^x g(s)\mathrm{d}s,$$

from where d'Alembert's formula follows, after finding F and G and plugging $x+ct$ and $x-ct$ in them respectfully instead of x. ■

Exercise 2. For u to be a classical solution to the wave equation it must be a $\mathcal{C}^{(2)}$ function, which immediately implies that f must be also $\mathcal{C}^{(2)}$. The initial velocity g, however, is included in the solution under the integral sign, which increases the regularity (say, for any continuous g, $\int_0^x g(s)\mathrm{d}s$ is a $\mathcal{C}^{(1)}$ function with the derivative $g(x)$, which is one of the forms of the fundamental theorem of calculus), therefore it is enough to request that $g \in \mathcal{C}^{(1)}$. ■

Exercise 3. Recall that a mathematical problem is well posed if the solution to this problem exists, it is unique, and it depends continuously on the initial data and parameters.

The existence of solutions is proved by presenting an explicit solution — d'Alembert's formula. Since we know the general solution to the wave equation, and, using the initial conditions, the only choice for F and G exactly as in d'Alembert's formula, it also proves the uniqueness.

To prove the continuous dependence, let me be a little more specific. First, in words, I want to show that if I have two initial value problems for the wave equation, with *close* initial conditions, then the solutions will stay *close* for all the future times. Here, of course, I need to define, what it means to be close for two functions. There are different ways to define closeness for the functions, I will chose the following one: I will say that functions p and q are ε close if they defined on the same set X and, moreover,

$$\max_{x\in X}|p(x)-q(x)|\leq\varepsilon.$$

Now consider the wave equation $u_{tt}=c^2u_{xx}$ with two initial conditions:

$$u(0,x)=f_1(x),\quad u_t(0,x)=g_1(x),$$

and

$$u(0,x)=f_2(x),\quad u_t(0,x)=g_2(x),$$

such that f_1,f_2 are ε_1 close on $\mathbf{R}$ and g_1,g_2 are ε_2 close on $\mathbf{R}$. I need to show that the two solutions u_1 and u_2 are close in some sense for all $t>0$. Indeed, for both u_1,u_2 I can write down d'Alembert's formulas, and now I can estimate (using the properties of the absolute value and integral)

$$\begin{aligned}|u_1(t,x)-u_2(t,x)|&\leq\frac{|f_1(x+ct)-f_2(x+ct)|}{2}+\frac{|f_1(x-ct)-f_2(x-ct)|}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}|g_1(s)-g_2(s)|\mathrm{d}s\\&\leq\frac{\varepsilon_1}{2}+\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2c}\int_{x-ct}^{x+ct}\mathrm{d}s,\\&\leq\varepsilon_1+\varepsilon_2t.\end{aligned}$$

Note that the right hand side of inequality does not depend on x, therefore I showed that

$$\max_{x\in\mathbf{R}}|u_1(t,x)-u_2(t,x)|\leq\varepsilon_1+\varepsilon_2t,$$

i.e., in terms of my definition my two solutions are $\varepsilon_1+\varepsilon_2t$ close, and this number can be made as small as required for fixed t for sufficiently small ε_1 and ε_2, which proves that the solution depends continuously on the initial conditions.

To summarize, this problem is *well posed.* ■